2 Prove that any algorithm which returns an ϵ-approximate minimizer of a function $f : [0,1]^n \to \mathbb{R}$ with $|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|_\infty$ requires at least $\left(\lfloor \frac{L}{2\epsilon} \rfloor\right)^n$ queries in the zero-th order oracle model if $\epsilon < L/2$.

Let $\epsilon < L/2$ and $I = [0,1]^n$. Suppose, to produce a contradiction, that an algorithm ALG returns an ϵ-approximate minimizer of a function $f : I \to \mathbb{R}$ with $|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|_\infty$ using less than $(\lfloor {}^{L}\!/_{2\epsilon} \rfloor)^n$ queries to a 0-th order oracle.

Suppose ALG chooses to evaluate initially at some $\boldsymbol{x}_1 \in I$. When given $f(\boldsymbol{x}_1) = 0$, ALG chooses $\boldsymbol{x}_2 \in I$, and so on, where ALG chooses $\boldsymbol{x}_i$ given $f(\boldsymbol{x}_k) = 0$ for all $k \in \{1, \ldots, i\}$ and $i \in \{1, \ldots, n\}$. Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ and call ALG's result $\overline{\boldsymbol{x}}$.

First, suppose that $\min_{\boldsymbol{x} \in X} \|\overline{\boldsymbol{x}} - \boldsymbol{x}\| \geq {}^{\epsilon}\!/_{L}$. Then the tightest bound we may obtain on $f(\overline{\boldsymbol{x}})$ is

$$\big|f(\overline{\boldsymbol{x}})\big| = \max_{\boldsymbol{x} \in X} \big|f(\overline{\boldsymbol{x}}) - f(\boldsymbol{x})\big| \leq L \min_{\boldsymbol{x} \in X} \|\boldsymbol{x} - \overline{\boldsymbol{x}}\| = M.$$

However we may have $M \geq \epsilon$. Thus, we may construct an f such that $f(\overline{\boldsymbol{x}}) \geq \epsilon$, which would yield

$$\epsilon = f(\overline{\boldsymbol{x}}) < f(\boldsymbol{x}^*) + \epsilon \leq f(\boldsymbol{x}_0) + \epsilon = \epsilon,$$

which is a contradiction.

Now, suppose that $\min_{\boldsymbol{x} \in X} \|\boldsymbol{x} - \overline{\boldsymbol{x}}\| < {}^{\epsilon}\!/_{L}$. Then, there must exist some point $\boldsymbol{x}'$ such that

$$\min_{\boldsymbol{x} \in X \cup \{\overline{\boldsymbol{x}}\}} \big\|\boldsymbol{x}' - \boldsymbol{x}\big\| \geq \frac{\epsilon}{L}. \tag{1}$$

To see this, we note that the set of points $\boldsymbol{p}$ satisfying $\|\boldsymbol{p} - \boldsymbol{x}\| \leq {}^{\epsilon}\!/_{L}$ is exactly the interior of an n-cube of side ${}^{2\epsilon}\!/_{L}$ centered at $\boldsymbol{x}$, which has an n-volume of $({}^{2\epsilon}\!/_{L})^n$. We know that $|X \cup \{\overline{\boldsymbol{x}}\}| \leq (\lfloor {}^{L}\!/_{2\epsilon} \rfloor)^n$, so the total number of n-cubes not satisfying Eq. (1) is at most $(\lfloor {}^{L}\!/_{2\epsilon} \rfloor)^n$. Since $\min_{\boldsymbol{x} \in X} \|\boldsymbol{x} - \overline{\boldsymbol{x}}\| < {}^{\epsilon}\!/_{L}$, the n-cube around $\overline{x}$ must overlap an n-cube around some point in X. Thus, the total n-volume V in the intersection of all of the n-cubes is strictly less than the sum of their n-volumes

$$V < ({}^{2\epsilon}\!/_{L})^n (\lfloor {}^{L}\!/_{2\epsilon} \rfloor)^n \leq 1.$$

Since V is the n-volume of the region of points that do not satisfy Eq. (1) and $V < 1$, the volume of I, there must exist some point $\boldsymbol{x}' \in I$ satisfying Eq. (1). Then as in the first case, the tightest bound on $|f(\boldsymbol{x}')|$ that we may obtain is $|f(\boldsymbol{x}')| \leq M$ for $M > \epsilon$. Thus, we may construct an f such that $f(\overline{\boldsymbol{x}}) = 0$ and $f(\boldsymbol{x}') = -\epsilon$, which would yield

$$0 = f(\overline{x}) < f(\boldsymbol{x}^*) + \epsilon \leq f(\boldsymbol{x}') + \epsilon = 0,$$

which is also a contradiction.

Therefore, any algorithm which returns an ϵ-approximate minimizer of a function f requires at least $(\lfloor {}^{L}\!/_{2\epsilon} \rfloor)^n$ queries to the oracle model, as desired.

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