
Math 389L
Problem Set 5
Friday, March 29, 2019

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ be independent Rademacher random variables (that is, $\boldsymbol{x}_i = \pm 1$ with equal probability) and $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ be a sequence of real numbers.

(a) Show that $\mathbb{E} e^{s\boldsymbol{x}_i} = \cosh(s)$.

(b) Prove that $\cosh(s) \leq e^{s^2/2}$.

(c) Use (a), (b), and Markov's inequality to prove that

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} a_i \boldsymbol{x}_i\right| \geq \epsilon\right) \leq 2e^{-\frac{\epsilon^2}{2\|\boldsymbol{a}\|_2^2}}.$$

Hint: Bound each tail $\mathbb{P}\big(\sum a_i \boldsymbol{x}_i \geq \epsilon\big)$ and $\mathbb{P}\big(\sum a_i \boldsymbol{x}_i \leq -\epsilon\big)$ separately and use a union bound. Try to use symmetry where possible.

(d) Conclude that the sum $\sum_{n=1}^{\infty} \frac{\boldsymbol{x}_n}{n}$ converges with probability 1 (i.e. almost surely). Remark why this is interesting.

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